δ-RINGS

Ethos of δ-rings

Fix a prime p and suppose we have a commutative ring A. Note that we have a Frobenius homomorphism $\phi : A/p \to A/p$. We would like to say something about lifts of this Frobenius map to A. Concretely, we would like to understand ring homomorphisms,

$$\psi : A \to A$$

such that $\psi(p) \subset (p)$ and the descended morphism $\overline{\psi} : A/p \to A/p$ coincides with ϕ.

Naturally, one might suggest that we try to understand these maps by understanding some kind of linearization of them. That is, we express $\psi(x)$ as $x^p + p\delta(x)$ and try to understand $\delta(x)$. This is the sense in which δ-rings are a kind of p-differentiation.

Definitions

Definition: A δ-ring is a pair (A, δ) where $\delta : A \to A$ is a map of sets which satisfies the following properties,

$$\delta(x + y) = \delta(x) + \delta(y) + \frac{1}{p}(x^p + y^p - (x + y)^p)$$
$$\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$$

We further require that $\delta(0) = \delta(1) = 0$.

These identities might seem arbitrary, but they fall out of the discussion above in the following sense: If (A, δ) is a δ-pair, then $\psi(x) = x^p + p\delta(x)$ defines a ring homomorphism $\psi : A \to A$.

The converse statement is not quite true. That is, given a lift of Frobenius ψ, it is not in general true that for $\psi(x) = x^p + p\delta(x)$, δ makes (A, δ) into a δ-pair. This is because we cannot deduce that δ satisfies the identities above without the assumption that A is p-torsion free (there is, however, an equivalence between δ-pairs and lifts of Frobenius if A is p-torsion free).

Note that given a δ-pair (A, δ), the induced lift of Frobenius ψ is δ-equivariant (also called a "δ-map"): $\delta(\psi(x)) = \psi(\delta(x))$. This is easy to check for A p-torsion free. The p-torsion case is trickier, but you essentially reduce to the p-torsion free case.

A very common example of δ-rings which will appear for us will be the Witt vectors of a characteristic p ring. We will see later that if we further assume our δ-rings are perfect (with respect to the induced lift of Frobenius), then Witt vectors are the only example (we'll define the ring of Witt vectors later in the lecture).

Note that we can define $\delta(n)$ for any integer n using the identities for δ-rings. One sees then that,

$$\delta(n) = \frac{n - n^p}{p}$$

This is really an artifact of the canonical ring homomorphism $\mathbb{Z} \to A$ being a δ-map for the unique δ-structure on $\mathbb{Z}$ (derived from the identity map being a lift of Frobenius).

In any case, one sees immediately that,

$$\delta(p^m) = p^{m-1}(1 - p^{mp-m})$$

The slogan in the literature is that δ lowers the order of p-vanishing by 1.

Using this, we can get the following nice lemma,

Lemma: If A is a δ-ring such that $p^n = 0$ for $n > 0$, then $A = 0$.

Proof: Since $n > 0$, A is naturally a $\mathbb{Z}_{(p)}$-algebra (using Bezout's lemma for gcd). The structure map $\mathbb{Z}_{(p)} \to A$ is a δ-map, and hence, using the above observation, we can continue the lower n so that eventually $n = 0$, i.e. a unit vanishes in A, which gives us the result.

A Quick Introduction to Witt Vectors

Let us now pivot to something which will help us later in the talk: Witt vectors.

Given a commutative ring A, we can examine the infinite product,

$$W(A) := A \times A \times ...$$

this has an obvious commutative ring structure, but it's not quite the one we want. Instead, we will use the obvious structure to induce a slightly more interesting structure. To do this, we will introduce the following polynomials in the formal variables $X_0, X_1, ...,$

$$W_n(X) := W_n(X_0, X_1, ...) = \sum_{i=0}^{n} p^i X_i^{p^{n-i}}$$

We then have a natural map,

$$\omega : W(A) \to A \times A \times ...$$

where $\omega(a) := \omega(a_0, a_1, ...) = (W_0(a), W_1(a), ...)$.

We let the codomain ring have the classical componentwise additive and multiplicative structure, and we ask for a ring structure on $W(A)$ which makes ω into a ring homomorphism. It turns out that there is a unique such ring structure, and we equip $W(A)$ with said ring structure. This will be our **ring of Witt vectors** $W(A)$.

It is, in fact, an endo-functor (I may have made up this term, but it means what you think it means) on the category of rings. In particular, for characteristic p rings A, we can transport the natural Frobenius homomorphism on A to an associated Frobenius morphism F on $W(A)$.

We mention here, should it be useful later, that there is another map,

$$V : W(A) \to W(A)$$

which is merely a group homomorphism (at least when A is characteristic p) such that $V(a_0, a_1, ...) = (0, a_0, a_1, ...)$.

Furthermore, $FV(a) = VF(a) = pa$.

Finally, we note that there is the notion of truncated Witt vectors. The case of most interest to us will be the 2-truncated Witt vectors. As sets, we define,

$$W_2(A) := A \times A$$

and we define the following ring structure,

$$\begin{aligned}(a,b) \oplus (c,d) &= (a+c, b+d+\frac{1}{p}(b^p + d^p - (b+d)^p))\\ (a,b) \otimes (c,d) &= (ac, b^p d + d^p b + pbd)\end{aligned}$$

Note that the second components of the sums and products resemble the sum and product formulas for δ-rings. In fact, the sections $s : A \to W_2(A)$ of the canonical projection to the first coordinate,

$$\pi_1 : W_2(A) \to A$$

are in bijection with the δ-structures on A.

We leave out some details about how the 2-truncated Witt vectors are the fiber product of A with itself when A is p-torsion free, and how we can use this to see that δ-structures on A are the same thing as derived Frobenius lifts on A. See the original Bhatt-Scholze paper for more details.

Properties of the Category of δ-Rings

We have a natural category of δ-rings, where morphisms are ring homomorphisms which are also δ-maps. It also has a forgetful functor to the usual category of rings. We can go further and show that this category of δ-rings has all limits and colimits. We will present this fairly informally.

It is natural to define a δ-structure on the limit of a diagram of δ-rings, using that fact that each individual homomorphism in the diagram is itself a δ-map. This δ structure will be naturally compatible with the projections to the individual δ-rings in the diagram.

For colimits, the situation is trickier. However, we may appeal to the equivalence between sections of $W_2(A) \to A$ and δ-structures to give a clean proof.

In any case, by some adjoint functor theorem, we have that the forgetful functor from δ-rings to rings has both left and right adjoints. The right adjoint will be the Witt vector functor.

The left adjoint of the forgetful functor, call it $\mathcal{L}$, when applied to the free ring over the set S will give us a 'free δ-ring', denoted $\mathbb{Z}\{S\}$. Notable we have,

$$\mathbb{Z}\{x\} = \mathbb{Z}[x_0, x_1, ...]$$

with

$$\delta(x_i) = x_{i+1}$$

Notice that since $\mathcal{L}$ is a left-adjoint, it is right exact, and hence given a δ-ring, we can 'forget it', establish a surjection from a free ring, and then apply $\mathcal{L}$ to get a surjection from a free δ-ring, which is always p-torsion free. Thus, we can systematically reduce certain problems about general δ-rings to problems about p-torsion free δ-rings.

We can get generators and relations for δ-rings via pushouts as well.

Example:

Suppose we have a polynomial $f(x, y) \in \mathbb{Z}[x, y]$, and we want to construct some version of $\mathbb{Z}\{x, y\}/(f(x, y))$. Essentially the way we do this is by forming a pushout (which exists in our category) as follows,

$$\begin{tikzcd} \mathbb{Z}[z] \arrow[r, "F"] \arrow[d, "G"'] & \mathbb{Z}\{x,y\} \arrow[d] \\ \mathbb{Z}\{t\} \arrow[r] & \mathbb{Z}\{x,y\}/(f(x,y))^{\delta} \end{tikzcd}$$

where $F(z) = f(x, y)$ and $G(z) = 0$. It is not hard to see that this object behaves as you would like, however, in general it is very hard to compute.

We can also form localizations of δ-rings so long as the multiplicative subset is 'ϕ-stable', i.e. $\phi(S) \subset S$ where S is the multiplicative subset of interest, and ϕ is the associated Frobenius. The δ-structure we get on $S^{-1}A$ will be compatible with the δ-structure on A.

The proof that these δ-structures exist on the localization is not difficult in the p-torsion free case, and in general we can reduce to that case by using free δ-rings.

Something that will maybe be useful later when we construct the prismatic site is the following,

Lemma: If A, B are p-adically complete p-torsion free rings, and $f : A \to B$ is an etale morphism, then for any δ-structure on A, there is a unique δ-structure on B making f a δ-map.

Proof Idea: Since the rings are p-adically complete, we can proceed mod p^N for all $N > 0$. The case of $N = 1$ is well-known, and for higher N, we use that the etale sites mod p^N are the same for any positive integer N (essentially, the higher N are merely thickenings of the mod p space, and so they are topologically the same).

We also note that if we have a δ-pair (A, δ) and $I \subset A$ an ideal with $\delta(I) \subset I$, then there is a compatible δ-pair $(A/I, \delta')$.

1. **Perfect δ-Rings**

Definition: A δ-ring A with induced lift of Frobenius ϕ is said to be perfect if ϕ is an isomorphism.

It turns out we can completely classify these rings,

Theorem: The following categories are equivalent,
(i) p-adically complete, perfect δ-rings
(ii) p-adically complete, p-torsion free rings that are perfect mod p
(iii) perfect char p rings

We will need two lemmas to justify these equivalences. The first of these is,

Lemma: If (A, δ) is a δ-pair and $px = 0$, then $\phi(x) = 0$.

At least one of the proofs of this statement is entirely formal, and can be found in Bhargav Bhatt's notes on δ-rings.

So, as per the lemma, the natural functor from (1) to (2) is the forgetful functor.
The second lemma we will need is,

Lemma: If A is a perfect $\mathbb{F}_p$-algebra, the cotangent complex $L_{A/\mathbb{F}_p}$ vanishes.

While we do not recall the exact definition/construction of the cotangent complex here, we mention that for $\mathbb{F}_p$-algebras A, we take a simplicial resolution of A over $\mathbb{F}_p$, compute the Kahler differentials, and tensor up to A. In some sense, we should think of the cotangent complex as the left-derived functor of the functor which assigns to each ring it Kahler differentials.

The point is, for finitely generated and free $\mathbb{F}_p$-algebras, the Frobenius acts trivially on the sheaf of differentials (because the differential brings down a p, which annihilates everything). On the other hand, the Frobenius is an isomorphism on A, and so it is an isomorphism on the whole cotangent complex.

Thus, we get an equivalence between the following categories,
(1) Perfect $\mathbb{F}_p$-algebras
(2) Flat $\mathbb{Z}/p^n$-algebras for fixed n, with reduction mod p being perfect
(3) p-adically complete, p-torsion free $\mathbb{Z}_p$-algebras with reduction mod p being perfect

The equivalence is deformation theory, using the vanishing of the cotangent complex. The equivalence of (2) and (3) is standard.

We leave out the details for the proof of the equivalence of the original 3 categories, see Bhatt's notes for a sketch.

REFERENCES (INFORMAL)

The references are the same as those listed on the website for the seminar, with the relevant sections used.