Prismatic cohomology - Distinguished elements and prisms

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These notes are based on Bhargav Bhatt's notes for his Eileberg lectures at Columbia, Kedlaya's notes on prismatic cohomology, and the Stacks project.

Fix a prime p. Recall that we defined a δ-ring to be a commutative ring A together with a map of sets $\delta : A \to A$ satisfying the necessary relations to make the map

$$\phi : A \to A, \quad \phi(a) = a^p + p\delta(a)$$

a morphism of commutative rings. Note that by construction, the morphism ϕ is automatically a lift of the Frobenius endomorphism in A/pA.

Remark (Zariski localization)**.** Given an affine scheme $\mathrm{Spec}(A)$ and a closed set $V(I) \subset \mathrm{Spec}(A)$, we define the localization of A along $V(I)$ to be the ring $S^{-1}A$, where $S = A \backslash \bigcup_{V(I)} \mathfrak{p}$, or equivalently, the set of all elements of A that are invertible on $V(I)$. Note that $I \in \mathrm{rad}(S^{-1}A)$ by construction.

We will implicitly use the following fact throughout the text: the localization of A along a closed subset has a unique δ-structure, making the morphism $A \to S^{-1}A$ a δ-map.

1 Distinguished elements

Definition. Let A be a δ-ring. An element $d \in A$ is called *distinguished* if $\delta(d)$ is a unit.

Let A be a δ-ring. Some useful properties of distinguished elements are the following:

Lemma. Assume $p, f \in \mathrm{rad}(A)$, and $u \in A^\times$. If f is distinguished, then so is uf.

Proof. We have

$$\delta(uf) = u^p\delta(f) + f^p\delta(u) + p\delta(u)\delta(f)$$

By assumption, the first summand on the right hand side is a unit, and the other two are in the Jacobson radical, so $\delta(uf)$ is a unit, and hence uf is distinguished. □

Lemma. Assume $p, f \in \mathrm{rad}(A)$, and $h = fg$. If h is distinguished, then f is distinguished and $g \in A^\times$.

Proof. We have

$$\delta(fg) = f^p\delta(g) + g^p\delta(f) + p\delta(f)\delta(g)$$

By assumption, the first and third summands on the right hand side are in the Jacobson radical, and $\delta(fg)$ is a unit, so $g^p\delta(f)$ must be a unit, hence proving both claims. □

Lemma. Assume $p, f \in \mathrm{rad}(A)$. Then f is distinguished if and only if $p \in (f, \phi(f))$.

Proof. If f is distinguished, $\delta(f)$ is a unit, then $\phi(f) = f^p + p\delta(p)$ implies $p \in (f, \phi(f))$.

Conversely, write $p = af + b\phi(f)$, and assume $\delta(f)$ is not a unit. Replacing A with its localization along $V(p, f, \delta(f))$, we may assume that $\delta(f) \in \mathrm{rad}(A)$. Using the expression of ϕ in terms of δ we get $p(1 - b\delta(f)) = f(a - bf^p)$. Using the previous lemmas, since $\delta(f) \in \mathrm{rad}(A)$, $1 - b\delta(f)$ is a unit so the left side is distinguished, and since $f \in \mathrm{rad}(A)$, it must be distinguished, contradicting the fact that $\delta(f) \notin A^\times$. □

Corollary. Assume $I \subset A$ is a Zariski locally principal ideal such that $(p, I) \subset \mathrm{rad}(A)$. Then the following are equivalent:

(1) $p \in (I, \phi(I))$

(2) There exists a faithfully flat map $A \to A'$ of δ-rings such that $IA' = (f)$ for a distinguished element f with $(p, f) \in \mathrm{rad}(A')$.

If these conditions are satisfied, then moreover we have $p \in (I^p, \phi(I))$.

Sketch of proof. Choose elements $(g_1, \ldots, g_r) = A$ such that IA_{g_i} is principal. Let $B = \prod_i A_{g_i}$, and let A' be the localization of B at $V(p, f)$. Then $A \to A'$ is fully faithful and $IA' = (f)$. The converse follows by the previous lemma. □

2 Derived completions

The theory of classical completions along an ideal I is not very well behaved in this context, since the rings that we will encounter are often not noetherian nor finitely generated. In fact, the classical completion functor is not right exact and it does not preserve flatness. These defects can be corrected by passage to the derived setting.

Definition. Lat A be a ring, and $I \subset A$ a finitely generated ideal. We say that a (derived) A-module M is derived I-complete if it satisfies one of the following equivalent conditions:

- For each $f \in I$, $\mathrm{RHom}_A(A_f, M) \simeq 0$.
- For each $f \in I$, $\mathrm{Rlim}(\cdots \xrightarrow{f} M \xrightarrow{f} M \xrightarrow{f} M) \simeq 0$.
- For each $f \in I$, the natural map $M \to \mathrm{Rlim}_n(M \otimes^L_{\mathbb{Z}[x]} \mathbb{Z}[x]/(x^n))$ is an isomorphism, where M is viewed as a $\mathbb{Z}[x]$ -module via the map $x \mapsto f$.

Remark. Note that a classical I-complete modules M is also derived I-complete. In fact, using the free resolution

$$0 \to A[T] \xrightarrow{1-Tf} A[T] \xrightarrow{T \mapsto f} A_f,$$

we get that the derived module $\mathrm{RHom}_A(A_f, M)$ is represented by the complex

$$\prod_{\mathbb{N}} M \xrightarrow{d} \prod_{\mathbb{N}} M$$

with the differential $d : (m_i) \mapsto (m_i - fm_{i+1})\}$. If M is I-complete we can find an inverse $s : (a_i) \mapsto (a_i + fa_{i+1} + f^2 a_{i+2} + \cdots)$. Therefore this complex is trivial in $D(A)$.

On the other hand, a derived I-complete module M will be classically complete if it is also I-adically separated. In particular, the classical and derived notions agree when the module A is noetherian and M is finitely generated.

The collection of derived I-complete A-modules forms a full triangulated subcategory of $D(A)$. The inclusion has a left adjoint $M \mapsto \widehat{M}$, which can be computed as follows: if $I = (f_1, \dots, f_r)$, then

$$\widehat{M} = \mathrm{Rlim}_n(M \otimes^L_{\mathbb{Z}[x_1,\dots,x_n]} \mathbb{Z}[x_1, \dots, x_n]/(x_1^n, \dots, x_r^n))$$

Lemma (Derived Nakayama)**.** A derived I-complete $M \in D(A)$ is 0 is and only if $M \otimes^L_A A/I \simeq 0$.

The derived complete modules we will work with will often be classically complete, as a consequence of the following lemma.

Lemma. If an A-module M has bounded f^∞-torsion, the derived f-completion of M is discrete and coincides with its classical f-completion.

The derived I-completion of a flat A-module M is I-completely flat, in the sense that $M \otimes^L_A A/I$ has cohomology only in degree 0, where it is given by a flat A/I-module.

3 Prisms

Definition. A *prism* is a pair (A, I) of a δ-ring A and and ideal $I \subset A$ such that A is (p, I)-derived complete and $p \in (I, \phi(I))$. We say that a prism is

- *perfect* if A is perfect.
- *bounded* if A/I has bounded p^∞-torsion.
- *crystalline* if $I = (p)$.

A map of prisms is simply a δ-map $A \to B$ that carries I into J.

Remark. **(Geometric interpretation)** One can think of a prism as a scheme $\mathrm{Spec}(A)$ together with a Cartier divisor $Z \subset \mathrm{Spec}(A)$ corresponding to $V(I)$. The condition $p \in (I, \phi(I))$ corresponds to requiring that Z and $\phi^{-1}Z$ meet only in characteristic p. In fact:

$$V(I) \cap \phi^{-1}V(I) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathfrak{p} \supseteq (I, \phi(I))\},$$

so $p \in (I, \phi(I))$ if and only if $V(I) \cap \phi^{-1}V(I) \subseteq V(p) \simeq \mathrm{Spec}(A/pA)$.

Example. Some basic examples of prisms are:

- *(Crystalline cohomology)* The pair $(\mathbb{Z}_{(p)}, (p))$ is a crystalline prism. In general, $(A, (p))$ is a crystalline prism for any p-torsion free p-adically complete δ-ring A.
- *(q-de Rham cohomology)* The pair $(\mathbb{Z}_p[[q-1]], [p]_q)$, where $[p]_q = \frac{q^p-1}{q-1}$, with δ-structure determined by $\phi(q) = q^p$.

- *(Breuil-Kisin cohomology)* The pair $(W[[u]], (d))$ for $W \subset \mathcal{O}_K$ the maximal unramified subring of the ring of integers of a p-adic field $K/\mathbb{Q}_p$, and d any element in the kernel of the surjection $W[[u]] \to \mathcal{O}_K$ that sends $u \mapsto \varpi$.
- *(A_∞-cohomology)* The pair $(\mathbb{Z}_p[q^{-p^\infty}]^\wedge_{(p,q-1)}, [p]_q)$ is a bounded prism,

For each of the previous pairs, the de Rham cohomology of a formally smooth A/I-scheme X admits a certain canonical deformation to A given by the cohomology theory named above.

Lemma. Let (A, I) be a prism. Then $\phi(I)A$ is principal and any generator is a distinguished element. In particular, if (A, I) is perfect, then $I = (f)$ for a distinguished element f.

Idea of the proof. Write $p = a + b$ with $a \in I^p$, $b \in \phi(I)$. Choose a faithfully flat map $A \to A'$ where $IA' = (f)$ is principal with f distinguished, as in the above corollary. In A' we have $b = y\phi(f)$ for some $y \in A'$. The strategy is then to show that y is a unit, which can be done using a similar argument to the one in the proof of (f dist. $\iff$ $p \in (f, \phi(f))$). We conclude by noticing that bA' principal implies bA principal by faithful flatness. □

Lemma. Let $(A, I) \to (B, J)$ be a map of prisms. Then $I \otimes_A B \simeq J$, and in particular $IB = J$. Conversely, if B is a derived (p, I)-complete δ-A-algebra, then (B, IB) is a prism iff $B[I] = 0$.

Thus, the forgetful functor $(B, IB) \mapsto B$ from prisms over (A, I) to δ-A-algebras is fully faithful.

Idea of the proof. Choose a faithfully flat δ-map $A \to A'$ such that $IA' = (f)$, and let B' be the localization of $A' \otimes B$, plus a furthen ind-Zariski localization to ensure $JB' = (g)$. Then $IB' \subseteq JB'$ is an inclusion of principal ideals generated by distinguished elements, and hence an equality by irreducibility. Since $B \to B'$ is faithfully flat, $IB \subseteq J$ must be an equality to.

Conversely, we have an exact sequence

$$0 \to B[I] \to I \otimes_A B \to IB \to 0$$

Both assertions $B[I] = 0$ and (B, IB) being a prism are equivalent to the fact that the map $I \otimes_A B \to IB$ is an isomorphism, hence concluding the claim. □