THE PRISMATIC SITE

We fix some notation that we will adhere to for the rest of the talk unless stated otherwise.

Let R be a formally smooth A/I-algebra. Recall that one of our goals with prismatic cohomology will be to find a complex of A-modules $\Delta_{R/A}$ with a Frobenius endomorphism $\phi_{R/A}$ such that,

1. $\Delta_{R/A} \hat{\otimes}^{\mathbb{L}}_A A/I$ is related to differential forms on R (relative to A/I)
2. $(\Delta_{R/A}[\frac{1}{I}], \phi_{R/A})$ is related to the p-adic etale cohomology of $R[\frac{1}{p}]$.

Essentially, we want try construct a sheaf of complexes of A-modules on A which interpolates between differential forms on the vanishing locus of I, and p-adic etale cohomology on the complement.

For the talk, we will assume (A, I) is bounded (i.e. A/I has bounded p-torsion), and (unless otherwise stated) $I = (d)$.

We are now in a position to define the (naive) *prismatic site* of R relative to A.

Definition:

$(R/A)_{\Delta}$ is the category whose objects are prisms (B, IB) over (A, I) with an A/I-algebra map $R \to B/IB$. The morphisms of this category are morphisms of prisms $(B, IB) \to (C, IC)$ over (A, I) such that the induced map $B/IB \to C/IC$ commutes with the structure maps from R, i.e.,

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & B/IB \arrow[d] \\ & C/IC \end{tikzcd}$$

commutes.

We denote objects of this category by $(R \to B/IB \leftarrow B)$.

We want to equip this category with the indiscrete Grothendieck topology, in which the only covering families are isomorphisms. This gives us the naive prismatic site, and all presheaves on this category are already sheaves.

Important Remark:

The *actual* naive prismatic site will be the opposite of this category. This has to do with us looking at rings instead of Spec of rings. We will stick with our incorrect category and ignore the issue.

We can now define one of the main sheaves on this site, which will be the functor,

$$\mathcal{O}_{\Delta} : (R/A)_{\Delta} \to \mathcal{C}$$

by $\mathcal{O}_{\Delta}(R \to B/IB \leftarrow B) = B$

Here, $\mathcal{C}$ is the category of (p, I)-complete $\delta - A$-algebras.

We can also define similarly

$$\overline{\mathcal{O}}_{\Delta} : (R/A)_{\Delta} \to \mathcal{C}'$$

by $\mathcal{O}_{\Delta}(R \to B/IB \leftarrow B) = B/IB$.

Here, $\mathcal{C}'$ is the category of p-complete R-algebras.

Note that we have, as sheaves,

$$\overline{\mathcal{O}}_{\Delta} \cong \mathcal{O}_{\Delta}/I\mathcal{O}_{\Delta}$$

Remarks:

1. There is another Grothendieck topology that we could have used on $(R/A)_{\Delta}$. Namely, we could require that $(B, IB) \to (C, IC)$ corresponds to a covering iff. it is I-*completely faithfully flat* (i.e. C/IC is a faithfully flat B/IB-module and $\mathrm{Tor}^B_{>0}(B/IB, C) = 0$). This topology induces genuinely different topoi. However, the cohomology complexes are the same. This is because we have an inclusion of sites,

$$((R/A)_{\Delta}, \text{Chaotic}) \to ((R/A)_{\Delta}, \text{flat})$$

If we have a sheaf on the right, then it is automatically a sheaf on the left, and so a resolution by injectives/flasques on the right is again a resolution by injectives/flasques. On the other hand, if you have a resolution of a presheaf/sheaf on the left, sheafification will give an injective/flasque resolution on the right, and since global sections of the sheafification agree with global sections of the presheaf, the two resolutions compute the same cohomology (we will discuss and define cohomology below in more detail).

2. For X a formally smooth A/I-scheme, we can define $(X/A)_{\Delta}$ in a similar way, where we probe again by prisms, but now we equip the flat topology as above. For this lecture, we will only be concerned with the affine case.

Basic Example: If $R = A/I$, then $(R/A)_{\Delta}$ has an initial object, so it will turn out that any cohomology you take will be trivial (concentrated in degree 0 as the global sections).

Cohomology

Notice that $\mathcal{C} := (R/A)_{\Delta}$ is a small category, and so for very general reasons, $\mathrm{Sh}(\mathcal{C}) = \mathrm{PSh}(\mathcal{C})$ has enough injectives. So, let us define a functor,

$$\Gamma : \mathrm{PSh}(\mathcal{C}) \to \mathrm{Ab}$$

(or into some other enhanced category) via

$$\mathcal{F} \mapsto \lim_{X \in \mathcal{C}} F(X) =: H^0(\mathcal{C}, \mathcal{F})$$

This functor is left exact, so we can take the derived functor $R\Gamma(\mathcal{C}, \cdot)$, which is really $R\lim_{X\in\mathcal{C}} \mathcal{F}(X)$.

If we do this for $\mathcal{O}_{\Delta}$ and $\overline{\mathcal{O}}_{\Delta}$, we get complexes,

$$\Delta_{R/A} := R\Gamma((R/A)_{\Delta}, \mathcal{O}_{\Delta}) \in \mathcal{D}(A)$$

and

$$\overline{\Delta_{R/A}} := R\Gamma((R/A)_{\Delta}, \overline{\mathcal{O}})_{\Delta} \in \mathcal{D}(R)$$

We note that $\Delta_{R/A}$ is a (p, I)-complete commutative algebra object, whereas $\overline{\Delta}_{R/A}$ is a p-complete commutative algebra object.

The lift of Frobenius ϕ on $\mathcal{O}_{\Delta}$ induces a ϕ-semilinear Frobenius map $\Delta_{R/A} \to \Delta_{R/A}$.

We note here the following remarkable fact,

$$\Delta_{R/A} \otimes^{\mathbb{L}}_A A/I \cong \overline{\Delta}_{R/A}$$

This really has to do with I being a Cartier divisor. If, for example, $I = (d)$, we have a finite resolution of A/I, and then for the sake of cohomology, the above fact is really a statement about $R\lim$ commuting with finite limits. We note that a similar statement should hold if I consisted of a regular sequence for A, in which case we have a finite resolution by Koszul. However, since all examples of note are captured by I being

Cartier, we restrict ourselves to that situation. (Thank you to Ivan Zelich for pointing this out to the author of this particular set of notes).

More on Cohomology

Here we discuss some more how one could, in theory, compute cohomology on sites with the indiscrete Grothendieck topology. The key is the following lemma,

Lemma: Let $\mathcal{C}$ be a small category with finite non-empty limits and a weakly final object (i.e. an object $X \in \mathcal{C}$ such that for all $Y \in \mathcal{C}$, $\mathrm{Hom}(Y, X)$ is nonempty). Then, $R\Gamma(\mathcal{C}, \mathcal{F})$ is quasi-isomorphic to the chain complex associated to the following cosimplicial object,

$$\mathcal{F}(X) \rightrightarrows \mathcal{F}(X \times X) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \mathcal{F}(X \times X \times X) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \ldots$$

We surpress a proof of this lemma in these notes, but try to provide some intuition. Note that, if instead of a *weakly* final object, we had instead a final object, then the cohomology would simply be $\mathcal{F}(X)$ concentrated in degree 0, because once one applies the sheaf to any object, it receives a unique map from $\mathcal{F}(X)$. While we lose uniqueness of maps from $\mathcal{F}(X)$, we get some kind of uniqueness of maps if we consider $\mathcal{F}$ applied to the whole Cech nerve (i.e. the above cosimplicial object).

Now, we would like to use this lemma to "compute" the prismatic complex and/or the Hodge-Tate complex. To do this, we need to check that the prismatic site has finite non-empty products and a weakly final object. In order to do this, we need the following extremely important lemma,

Lemma: Consider the forgetful functor from the category of prisms over (A, I) to the category of δ-pairs over (A, I). This forgetful functor admits a left adjoint, called the *prismatic envelope*.

Before we explain the proof, let us spell out, in more detail, the statement of the lemma. In effect, what it says is that given a δ-pair (B, J) over (A, I), there is a prism (C, IC) over (A, I) such that there is a universal morphism of δ-pairs over (A, I), $(B, J) \to (C, IC)$. Universal here means that for any other prism (D, ID) over (A, I), any morphism of δ-pairs from (B, J) to (D, ID) factors through a morphism of prisms $(C, IC) \to (D, ID)$ over (A, I).

The proof itself is not enlightening, but we present it nevertheless for the sake of completion.

Proof: Let us assume, for simplicity, that $I = (d)$. Given such a δ-pair (B, J), in order to construct such a C, we note that the ideal for C will have to be IC, and it will need to admit a map $J \to IC$. So, we could naively try to construct C by "adding" elements of the form $\dfrac{x}{d}$ to B where $x \in J$. If we change adding to formally adjoining (when considered as δ-pairs), we get a ring B'. Let B_1' be the largest quotient of B' which is d-torsion free. Let B_1 then be the (p, d)-completion of B_1'.

If B_1 were d-torsion free, then (B_1, dB_1) would suffice. If not, then iterate this operation transfinitely to arrive at a $\tilde{B}$. This ring is the countably-filtered colimit of (p, d)-complete rings, and hence is (p, d)-complete ("κ-filtered colimits commute with κ-small limits"). Thus, we get (C, dC), and one checks that this satisfies the necessary properties.

We are now in a position to show that the naive prismatic site has all finite nonempty coproducts (recall that what we think of as the naive prismatic site is the opposite of the actual naive prismatic site).

Indeed, suppose we have two objects $(R \to B/IB \leftarrow B)$ and $(R \to C/IC \leftarrow C)$. We have coproducts in the category of δ-rings, and so we have form $D_0 = B \otimes_A C$. Note that we have the following square,

$$\begin{tikzcd} B/IB \arrow[r] & D_0/ID_0 \\ R \arrow[u] \arrow[r] & C/IC \arrow[u] \end{tikzcd}$$

which is not necessarily commutative. This suggests that ID_0 is not the correct ideal, and we should look at something larger. There is an ideal J which works, so we have a δ-pair (D_0, J) over (A, I). We can take its prismatic envelope (D, ID), and one checks that this gives the desired coproduct (see Prof. Bhatt's notes for more details).

Now, we wish to exhibit a weakly initial object for the naive prismatic site. Let F_0 be a free $\delta - A$-algebra with a surjection $F_0 \to R$, and with kernel J. Then, (F_0, J) is a δ-pair over (A, I). We can then take the prismatic envelope of (F_0, J) to get (F, IF). One then checks that $(R \to F/IF \leftarrow F)$ is a weakly initial object. Thus, we can compute prismatic cohomology as cohomology of,

$$\mathcal{F}(F) \rightrightarrows \mathcal{F}(F \times F) \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \mathcal{F}(F \times F \times F) \substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow} \ldots$$

Notice that since taking prismatic envelopes in general requires taking transfinite iterations, it is not practical to use the above complex to actually compute cohomology. Rather, much like Cech cohomology, one uses it to prove things about prismatic cohomology (we'll see an example of this, but we won't provide the proof).

Hodge-Tate Comparison

Recall that if we have a ring map $B \to C$, then we have the deRham complex,

$$\Omega^*_{C/B} := \{C \xrightarrow{d} \Omega^1_{C/B} \xrightarrow{d} \Omega^2_{C/B} \xrightarrow{d} \ldots\}$$

which is a strictly graded commutative B-dga via wedge products (differential satifies *signed* Leibniz).

We have a nice universal property for this complex via the universal property for Kahler differentials and the universal property for the exterior algebra:

Theorem: If (E^*, d) is a graded commutative B-dga such that $E^i = 0$ for $i < 0$, and there is a map $\eta : C \to E^0$ satisfying $d(\eta(x))^2 = 0$ for all $x \in C$ (which is always true if E is strictly graded commutative), then there is a unique extension of η to a map of B-dgas $\Omega^*_{C/B} \to E^*$.

We would like to turn the Hodge-Tate complex into a dga, and to do that we need a differential. That is afforded to us by considering the following exact sequence of (pre)sheaves on the prismatic site (assume, for now, that $I = (d)$):

$$0 \to \mathcal{O}_{\mathbb{\Delta}}/d \xrightarrow{\cdot d} \mathcal{O}_{\mathbb{\Delta}}/d^2 \to \mathcal{O}_{\mathbb{\Delta}}/d \to 0$$

where the second map is the canonical projection map. Taking cohomology gives a Bockstein differential,

$$\beta_d : H^i(\overline{\mathbb{\Delta}}_{R/A}) \to H^{i+1}(\overline{\mathbb{\Delta}}_{R/A})$$

It is apparently a standard fact that these differentials form a chain complex.

Note we also have a canonical map $\eta : R \to H^0(\overline{Prism})$. We then check that $(H^*(\overline{\mathbb{\Delta}}_{R/A}), \beta_d)$ is an A/I-dga and η satisfies $\beta_d(\eta(x))^2 = 0$ (this is only nontrivial for $p = 2$ because then $H^*(\overline{\mathbb{\Delta}}_{R/A})$ can have 2-torsion, in which case it need not be strictly graded commutative). The proof in the $p = 2$ case is attributed to Prof. de Jong (it can be found in Prof. Bhatt's notes), and it uses the quasi-isomorphism between $R\Gamma((R/A)_{\Delta}, \overline{\mathcal{O}}_{\Delta})$ and the cosimplicial object obtained by applying $\overline{\mathcal{O}}_{\Delta}$ to the Cech nerve.

So, we have a morphism of A/I-dgas,

$$\tilde{\eta} : (\Omega^*_{R/(A/I)}, d) \to (H^*(\overline{\Delta}_{R/A}), \beta_d)$$

which we call a Hodge-Tate comparison map. Later, we will show that this map is actually an isomorphism.

Remark 1

For the moment, let us return to the issue of I not necessarily being principal, but Cartier.

The idea is to consider "Breuil-Kisin" twists. Concretely, we denote, for any A/I-module M, $M\{n\} := M \otimes_{A/I} (I/I^2)^{\otimes n}$ (because I is Cartier, this is defined even for $n < 0$). We then have the following exact sequence,

$$0 \to I^{n+1}\mathcal{O}_{\Delta}/I^{n+2} \to I^n\mathcal{O}_{\Delta}/I^{n+2} \to I^n\mathcal{O}_{\Delta}/I^{n+1} \to 0$$

Taking cohomology gives a map,

$$\beta_I : H^n(I^n\mathcal{O}_{\Delta}/I^{n+1}) \to H^{n+1}(I^{n+1}\mathcal{O}_{\Delta}/I^{n+2})$$

where cohomology is taken as $\mathcal{O}_{\Delta}$-modules. Now,
$H^n(I^n\mathcal{O}_{\Delta}/I^{n+1})$
$= H^n(I^n\mathcal{O}_{\Delta} \otimes^{\mathbb{L}}_A A/I)$
$= H^n(\mathcal{O}_{\Delta} \otimes^{\mathbb{L}}_A I^n \otimes^{\mathbb{L}}_A A/I)$
$= H^n(\mathcal{O}_{\Delta} \otimes^{\mathbb{L}}_{A/I} I^n/I^{n+1})$
$= H^n(\overline{\mathcal{O}}_{\Delta})\{n\}$
(there are hidden details in the lines above; for example, some things should really be viewed as complexes in a derived category...)

Thus, we get $\beta_I : H^n(\overline{\mathcal{O}}_{\Delta})\{n\} \to H^{n+1}(\overline{\mathcal{O}}_{\Delta})\{n+1\}$

which is a Bockstein differential as before. We then get the same kind of comparison map as we did for $I = (d)$.

Remark 2

In the future, we really want to work with p-complete things, so we define $\hat{\Omega}^i{}_{R/(A/I)}$ to be the derived p-completion of the usual R-module. There is a similar universal property for degree 0 maps extending to maps of A/I-dgas in this completed context. The difference is that the target dga must also be p-complete. We then get a Hodge-Tate comparison map in this case as well.

REFERENCES (INFORMAL)

The references are the same as those listed on the website for the seminar, with the relevant sections used.