SPECTRA: DEFINITION AND EXAMPLES

SANG MIN KO

1. Introduction

1.1. **Spainer Whitehead category.** Let $\mathrm{Map}_*(X,Y)$ be the space of continuous functions from X to Y and let $[X,Y]=\pi_0(X,Y)$. One of the ultimate goals of homotopy theory is to describe $[X,Y]$. This is an extremely difficult problem because the set $[X,Y]$ does not have algebraic structure in general. However, if $X\cong\Sigma X'$ for some pointed space X', then $[X,Y]\cong\pi_1(\mathrm{Map}_*(X',Y))$ admits a group structure. Moreover, if $X\cong\Sigma^2X''$ for some pointed space X'', then $[X,Y]\cong\pi_2\mathrm{Map}_*(X'',Y)$ is abelian group.

One can use these observations to study $[X,Y]$ in general: There are maps

$$[X,Y]\to[\Sigma X,\Sigma Y]\to[\Sigma^2X,\Sigma^2Y]\to\cdots$$

The set $[X,Y]_s=\varinjlim_n[\Sigma^nX,\Sigma^nY]$ is called the group of stable maps from X to Y for finite CW complexes X and Y. By the Freudenthal suspension theorem, the colimit system stabilizes after a finite step and $[X,Y]_s$ provides an approximation to $[X,Y]$.

Theorem 1.1. *Let X be an n-connected topological space. Then*

$$\pi_k(X)\to\pi_{k+1}(\Sigma X)$$

is an isomorphism for $k\le 2n$.

The set $[X,Y]_s$ of stable maps can be regarded as a linearized version of the homotopy set $[X,Y]$. To extend this linearization, we can linearize the homotopy category of Top itself. The resulting category is known as the *stable homotopy category* or the *homotopy category of spectra.*

Consider a category $\mathcal{C}$ whose objects are finite pointed CW complexes, and whose morphisms are given by $\mathrm{Hom}(X,Y)=[X,Y]_s$. By construction, the suspension functor $X\mapsto\Sigma X$ gives fully faithful embedding from $\mathcal{C}$ to itself. However, the suspension functor is not an equivalence in general. To obtain a slightly larger category $\mathcal{C}^{SW}$ where the suspension functor is an equivalence, we formally invert Σ^nX for all $n\in\mathbb{Z}$. Conceretly, $\mathrm{Ho}(\mathrm{Sp}^{\mathrm{fin}})$ is a category where

(1) objects are pairs (X,n), where X is pointed CW complex and $n\in\mathbb{Z}$, which can be viewed as Σ^nX.
(2) morphisms are given by $\mathrm{Hom}_{\mathrm{Sp}^{\mathrm{fin}}}((X,n),(Y,m))=\varinjlim_k[\Sigma^{n+k}X,\Sigma^{m+k}Y]$.

This is called *Spanier-Whitehead category.* In fact, it is homotopy category of finite spectra.

1.2. **Generalized cohomology theory.** Stable homotopy theory is to study invariants of (pointed) topological space that are invariant under suspension. Singular cohomology is one example of stable invariant:

$$\tilde{H}^n(X) \cong \tilde{H}^{n+1}(\Sigma X)$$

More generally, one can consider generalized cohomology theories:

Definition 1.2. A (reduced) generalized cohomology theory is a sequence of functors $E^n : \mathrm{Ho}(\mathrm{Top}^{\mathrm{op}}_{\mathrm{CW},*}) \to \mathsf{Ab}$ for $n \in \mathbb{Z}$ with suspension isomorphism

$$\rho_n : E^{n+1}(X) \xrightarrow{\simeq} E^n(\Sigma X)$$

satisfying

(1) (exactness) For inclusion $\iota : A \to X$, consider a mapping cone $C\iota$ and we have cofiber sequence $A \to X \to C\iota$. Then it gives an exact sequence of abelain groups
$$E^n(C\iota) \to E^n(X) \to E^n(A)$$
(2) (additivity) For a collection of $\{X_i\}_{i\in I}$ of pointed CW complexes,
$$E^n(\bigvee_{i\in I} X_i) \cong \prod_{i\in I} E^n(X_i)$$

Brown representability theorem guarantees that a generalized cohomology theories E^n is representable by a pointed space X_n, and the suspension isomorphism ρ_n gives a homotopy equivalence $X_n \cong \Omega X_{n+1}$. In otherwords, any generalized cohomolgoy theory is representbale by a spectrum X.

2. Spectra

2.1. **Sequential spectrum.**

Definition 2.1. A (sequential) spectrum is a sequence of pointed spaces $X = (X_n)_{n\in\mathbb{N}}$ equipped with structure maps $\rho_n : \Sigma X_n \to X_{n+1}$ for each $n \in \mathbb{N}$. A morphism $f : X \to Y$ between spectra is a sequence of maps $f_n : X_n \to Y_n$ such that the following diagram commutes:

$$\begin{tikzcd}
\Sigma X_n \arrow[r] \arrow[d, "\rho_n^X"] & \Sigma Y_n \arrow[d, "\rho_n^Y"] \\
X_{n+1} \arrow[r] & Y_{n+1}
\end{tikzcd}$$

Definition 2.2. Let X be a spectrum. The k-th stable homotopy group $\pi_k(X)$ of X is defined as $\varinjlim_n \pi_{n+k}(X_n)$, where the map $\pi_{n+k}(X_n) \to \pi_{n+k+1}(X_n+1)$ is given by the composition

$$\pi_{n+k}(X_n) \to \pi_{n+k+1}(\Sigma X_n) \to \pi_{n+k+1}(X_{n+1})$$

Example 2.3. Let X be a pointed space. The suspension spectrum $\Sigma^\infty X$ is the spectrum whose n-th level is $\Sigma^n X$. In particualr, the sphere spectrum $\mathbb{S} = \Sigma^\infty S^0$. The stable homotopy group $\pi_k(\Sigma^\infty X)$ is given by

$$\pi_k(\Sigma^\infty X) = \varinjlim_n \pi_{n+k}(\Sigma^n X) = \pi_k^s(X).$$

2.2. **Ω-spectrum.**

Definition 2.4. A spectrum X is called Ω-spectrum if the adjoint map $\tilde{\rho}_n : X_n \to \Omega X_{n+1}$, corresponding to the structure map ρ_n, is a homotopy equivalence for all $n \in \mathbb{N}$. The zero-th space X_0 of Ω-spectrum is called an infinite loop space.

Brown representabiltiy implies that every generalized cohomology theories is represented by an Ω-spectrum. If X is Ω-spectrum, then $\pi_k(X) \cong \pi_{k+n}(X_n)$ for all n s.t. $k + n \geq 0$.

Example 2.5. Consider ordinary cohomology $H^n(-; A)$ with coefficients in an abelain group A. The sequence $\{H^n(-; A)\}_{n\in\mathbb{Z}}$ is represented by an Ω-spectrum HA. Specifically, $(HA)_n = K(A, n)$, where $K(A, n)$ is Eilenberg-Maclane space and $K(A, n) \simeq \Omega K(A, n + 1)$. The stable homotopy groups of HA are given by

$$\pi_k(HA) \cong \begin{cases} A & \text{if } k = 0 \\ 0 & \text{if } k \neq 0. \end{cases}$$

Example 2.6. Let $U = \varinjlim_n U(n)$ be the infinite unitary group and $BU = \varinjlim_n BU(n)$ be its classifying space. The space $\mathbb{Z} \times BU$ classifies stable complex vector bundles. The Bott periodicity theorem states that

$$\Omega(\mathbb{Z} \times BU) \simeq U \quad \text{and} \quad \Omega U \simeq \mathbb{Z} \times BU.$$

Complex K-theory is a generalized cohomology represented by an Ω-spectrum KU. Specifically, $(KU)_{2n} = \mathbb{Z} \times BU$ and $(KU)_{2n+1} = U$, with

$$\pi_k(KU) \cong \left\{ \begin{array}{lll} \mathbb{Z} & \text{if} & k = 2n \\ 0 & \text{if} & k = 2n+1 \end{array} \right.$$

2.3. **Thom spectrum.**

Definition 2.7. Let $E \to B$ be a vector bundle of rank n. The *Thom space* of E is defined as $\mathrm{Th}(E) := D(E)/S(E)$ where $D(E)$ is the disk bundle and $S(E)$ is the sphere bundle.

The Thom space of E is also denoted by B^E or $\Sigma^E B$. When E is trivial bundle of rank n over B, we have

$$\mathrm{Th}(E) \cong B \times D^n / B \times S^{n-1} \cong B \times S^n / B \times \{\infty\} \cong \Sigma^n(B_+)$$

The Thom space of E over B can be regarded as a twisted suspension of B by E.

Lemma 2.8. $\mathrm{Th}(E \oplus \underline{\mathbb{R}}^n) \simeq \Sigma^n \mathrm{Th}(E)$

Definition 2.9. The Thom spectrum functor $\mathrm{Th} : K_0(B) \to \mathrm{Sp}$ is defined by $\mathrm{Th}(W) = \Sigma^{-d}\mathrm{Th}(E)$ for a virtual bundle $W = E - \underline{\mathbb{R}}^d$

For example, let M be a closed manifold with an embedding $f : M \to \mathbb{R}^n$. Then $TM \oplus vM \cong \underline{\mathbb{R}}^n$ where vM is the normal bundle of f. Therefore,

$$\mathrm{Th}(-TM) = \Sigma^{-n}\mathrm{Th}(vM)$$

There is a universal Thom spectrum MO, whose base space is BO which classifies stable real vector bundle. $BO(n) = \mathrm{Gr}_n(\mathbb{R}^\infty)$ is the classifying space for $O(n)$, and the inclusion $O(n) \hookrightarrow O(n + 1)$ induces a canonical map $BO(n) \hookrightarrow BO(n + 1)$. Thus,

$$BO := \varinjlim_n BO(n)$$

is the classifying space for the stable orthogonal group $O := \varinjlim_n O(n)$. We have a tautological bundle $\gamma^n \to BO(n)$, and $MO(n) := \mathrm{Th}(\gamma^n)$. The following commutative diagram

$$\begin{tikzcd} \gamma^n \oplus \underline{R} \arrow[r] \arrow[d] & \gamma^{n+1} \arrow[d] \\ BO(n) \arrow[r] & BO(n+1) \end{tikzcd}$$

gives a structure map of the Thom spectrum MO:

$$\Sigma MO(n) \cong \mathrm{Th}(\gamma^n \oplus \underline{R}) \to \mathrm{Th}(\gamma^{n+1}) = MO(n+1)$$

The Pontryagin-Thom construction implies the following:

Theorem 2.10. *$\pi_d(MO(n))$ is isomorphic to the group of d-dimensional closed manifold up to cobordism.*

2.4. **Cellular spectrum.** A cell complex is a space that is built by attaching cells D^n via attaching maps. Cellular spectra are essentially cell complexes that may include negative-dimensional cells.

Definition 2.11. A spectrum X is *cellular* if each space X_n is a based cell complex, and the structure map $\Sigma X_n \to X_{n+1}$ is an inclusion of a subcomplex.

A *stable k-cell* in X, for $k \in \mathbb{Z}$, is $(k+n)$-cell in X_n for $n \geq 0$. This corresponds to a $(k+n+1)$-cell in $\Sigma X_n \subseteq X_{n+1}$.

Example 2.12. If K is a based cell complex, then $\Sigma^\infty K$ is cellular, with a stable k-cell for every k-cell other than the basepoint. For instance, $\mathbb{S}^d$ has a single stable d-cell.

Definition 2.13. A map $f : A \to X$ is called *relative cellular spectrum* if the n-th relative structure map

$$A_n \cup_{\Sigma A_{n-1}} \Sigma X_{n-1} \to X_n$$

is a relative cell complex for all $n \in \mathbb{N}$.

The follwoing theorem states that cellular spectra are also built by attaching stable cells via attaching maps, similar to how cell complex are built.

Theorem 2.14. *A map $f : A \to X$ is a relative cellular spectrum if and only if*

$$A = X^{(-1)} \to X^{(0)} \to \cdots \to X^{(i-1)} \to X^{(i)} \to$$

where $X = \varinjlim_n X^{(n)}$, and $X^{(i-1)} \to X^{(i)}$ is obtained by attaching cells:

$$\begin{tikzcd} \bigvee_k F_{n_k} S_+^{m_k - 1} \arrow[r] \arrow[d] & \bigvee F_{n_k} D_+^{m_k} \arrow[d] \\ X^{(i-1)} \arrow[r] & X^{(i)} \end{tikzcd}$$

The category of (sequential) spectra Sp has the *stable model structure*:

(1) The cofibrations are the retracts of the relative cell complex.
(2) The weak equivalences are the stable equivalences.

(3) The fibrations are the stable fibrations: maps $X \to Y$ such that $X_n \to Y_n$ is a Serre fibration, and the following square is a homotopy pullback diagram:

$$\begin{array}{ccc} X_n & \longrightarrow & \Omega X_{n+1} \\ \downarrow & & \downarrow \\ Y_n & \longrightarrow & \Omega Y_{n+1} \end{array}$$

References

1. J. F. Adams, *Stable homotopy and generalised homology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, Ill.-London, 1974. MR 402720
2. Jacob Lurie, *Higher algebra*, (2017).
3. Cary Malkiewich, *Spectra and stable homotopy theory (first 6 chapters, draft version)*, (2023).

Department of Mathematics, Columbia University
Email address: `smko77@math.columbia.edu`